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Reduced Weighted Complete Intersection and Derivations

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Let $A = \mathbb{F}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a zero-dimensional weighted complete intersection (char $\mathbb{F} = 0$). We prove a general result on the (homogeneous) derivations of A. In particular we deduce that in the "generic" case when $\mathbb{F}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1})$ is reduced ($\deg(f_i) \leq \deg(f_n), i = 1, \ldots, n-1$) there are no non-trivial derivations of strictly negative degree. If moreover all the weights of the variables x_i are even it follows that the Serre spectral sequence of any orientable fibration $F \hookrightarrow E \to B$, with $H^*F = A$ as graded algebras, collapses at the E_2-term, thus verifying a conjecture of Halperin. We also discuss several examples. © 1996 Academic Press, Inc.

INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathbb{F}$ be a field of characteristic zero and let A be a *weighted zero-dimensional complete intersection*, i.e., a commutative $\mathbb{F}$-algebra of the form

$$A = \mathbb{F}[x_1, \ldots, x_n]/I,$$

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where the ideal I is generated by a regular sequence of length n, $(f_1, \dots, f_n)$. Here the variables have (strictly positive, integral) weights, denoted by $|x_i| = w_i$, $1 \le i \le n$, and the equations are (weighted) homogeneous with respect to these weights. They are arranged for future convenience in the increasing order of the degrees: $0 < |f_1| \le \cdots \le |f_n|$. Consequently the algebra A is graded and one may speak about its homogeneous *degree k derivations* (any integer k). A degree k $\mathbb{F}$-linear map $\theta: A^* \to A^{*+k}$ belongs to $\mathrm{Der}^k(A)$ if $\theta(ab) = \theta(a)b + a\theta(b)$, for any $a, b \in A$. One of the most important open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees:

Halperin Conjecture. If A is as above, then $\mathrm{Der}^{<0}(A) = 0$.

Assuming that all the weights are even, this has the following topological interpretation. If a space X has $H^*(X; \mathbb{F}) = A$ (as graded algebras) then it is straightforward and classical to see that the vanishing of $\mathrm{Der}^{<0}(A)$ implies the collapsing at the E_2-term of the Serre spectral sequence with $\mathbb{F}$-coefficients of any orientable fibration having X as fiber. Actually the above vanishing and collapsing properties are equivalent for $\mathbb{F} = \mathbb{Q}$ and X a rational space, see e.g., [Me]. The Halperin Conjecture has been verified in several particular cases:

—equal weights ($w_1 = \cdots = w_n$), see [Z];

—$n = 2$ [Th];

—"fibered" algebras [M];

—homogeneous spaces of equal rank compact connected Lie groups ($A = H^*(G/K)$), see [ST].

If $\mathbb{F}$ is algebraically closed, we are going to verify it in the "generic" case when $\mathbb{F}[x_1, \dots, x_n]/J (J = (f_1, \dots, f_{n-1}))$ is *reduced.*

Our method will also provide information on $\mathrm{Der}^{\circ}(A)$, the infinitesimal form of the algebraic group $\mathrm{Aut}(A)$ (the degree zero graded algebra automorphisms of A). It is appropriate to mention here that the study of $\mathrm{Der}^*(A)$ turns out to be very useful also in the case of the coordinate rings of positive dimensional algebraic quasi-homogeneous singularities with good $\mathbb{F}^*$-action. In particular the vanishing of $\mathrm{Der}^{<0}$ has nice consequences from the point of view of moduli space theory of deformations, see, e.g., [W]. In our zero-dimensional case the common zero set of $f_1, \dots, f_n$ consists of the origin alone, the geometry is poor, and the difficulty increases. However, we can prove the following main result (see the next section):

THEOREM. *If $J = (f_1, \dots, f_{n-1})$ as above and $D \in \mathrm{Der}^*(\mathbb{F}[x_1, \dots, x_n])$ has the property that $D(f_i) \in J$, for any $i = 1, \dots, n$, then D vanishes modulo the radical $\sqrt{J}$.*

COROLLARY. *Assume that* $\mathbb{F}[x_1, \dots, x_n]/J$ *is reduced. Then*:

(i) $\mathrm{Der}^{<0}(A)$ *vanishes*.

(ii) *If moreover* $|f_{n-1}| < |f_n|$ *then the identity component of* $\mathrm{Aut}(A)$ *equals the algebraic* 1-*torus consisting of grading automorphisms* t ($t \in \mathbb{F}^*$ *acts on* A^p *as the scalar* $t^p, \forall_p$).

Any derivation $\theta \in \mathrm{Der}^k(A)$ is obviously induced by a derivation $D \in \mathrm{Der}^k(\mathbb{F}[x])$ which leaves the ideal I invariant, for any zero-dimensional weighted complete intersection $A = \mathbb{F}[x]/I$. This follows from the fact that one may construct derivations of a polynomial algebra with arbitrarily prescribed values on the algebra generators. If $k < 0$ this just means that $Df_i \equiv 0 \pmod{J}$ for any $i = 1, 2, \dots, n$, as in the hypothesis of our theorem (remember that f_n was chosen to have maximal degree among the equations). Moreover one also knows that $|x_i| \le |f_i|, 1 \le i \le n$, for any such algebra A, if the weights are also arranged in increasing order $|x_1| \le \cdots \le |x_n|$. This follows, with the same proof, from a weaker form of Lemma 2.5 of [FH], replacing the strong arithmetic condition by the arithmetic condition. Therefore if $k < 0, D$ will induce the trivial derivation of A is and only if $Dx_i \equiv 0 \pmod{J}, 1 \le i \le n$, again by degree inspection. Thus the conclusion of our theorem seems to be quite natural.

On the other hand the complex ($\mathbb{F} = \mathbb{C}$) *isolated* singularity weighted complete intersection curves $B = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_{n-1})$ as above are defined by the Zariski open condition "$z \neq 0$ and $f_1(z) = \cdots = f_{n-1}(z) = 0$ implies $d_z f_1, \dots, d_z f_{n-1}$ linearly independent." This implies that B is reduced [Lo, Proposition (1.10)], hence our "Zariski generic" terminology in connection with our corollary.

In the remaining two sections we shall first give a geometric proof of the above theorem and of the corollary, and second indicate a large class of examples of applications. In particular this will provide many-parameter families of isomorphism types of algebras A with $\mathbb{F}[x]/J$ reduced, as above (for a discrete series of such examples, namely $A = H^*(G/K), rkK = rkG$, and G simple, see also [S]).

1. PROOFS OF THE THEOREM AND THE COROLLARY

Proof of the corollary. As we have seen earlier, any derivation $\theta \in \mathrm{Der}^k(A)$ is induced by a derivation $D \in \mathrm{Der}^k(\mathbb{F}[x])$ which leaves the ideal I invariant. In (i) we know that $k < 0$, and hence $Df_i \equiv 0 \pmod{J}$ for any $1 \le i \le n$, as we have already remarked in the Introduction. In the case (ii) we are dealing with $k = 0$. We may see that necessarily $Df_i \equiv 0 \pmod{J}$, and $i < n$, and $Df_n = \lambda f_n \pmod{J}$ for some $\lambda \in \mathbb{F}$, by degree inspection, given the additional hypothesis $|f_{n-1}| < |f_n|$. Therefore we may replace in this case D by $D - (\lambda/|f_n|)e$, where e is the degree zero Euler derivation (which acts on A^p as the scalar p, any p, A being an arbitrary graded algebra). In this way all the requirements of our above theorem are again fulfilled. We may then invoke it to infer that $D \equiv 0$ modulo $\sqrt{J} = J \subseteq I$ in both cases and get that $\mathrm{Der}^{<0}(A) = 0$ in (i) and $\dim_{\mathbb{F}} \mathrm{Der}^{\circ}(A) = 1$ in (ii). Given the faithful correspondence between connected closed subgroups and Lie subalgebras in characteristic zero [H, pp. 87, 88], the assertion made in (ii) immediately follows from a dimension argument.

Proof of the theorem. Our hypotheses on D imply in particular that $D(J) \subseteq J$. The zero set Z of J is a complete intersection curve in $\mathbb{F}^n$, invariant under the good $\mathbb{F}^*$-action given by $t \cdot (x_1, \ldots, x_n) = (t^{w_1}x_1, \ldots, t^{w_n}x_n)$, for $t \in \mathbb{F}^*$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$. If one takes the primary decomposition of J, $J = \bigcap_{j=1}^r q_j$, with associated primes $p_j = \sqrt{q_j}$, $j = 1, \ldots, r$, then the irreducible components of Z are $Z_j =$ the zero set of p_j, $j = 1, \ldots, r$ (each of them being a $\mathbb{F}^*$-invariant irreducible curve in $\mathbb{F}^n$), and moreover $D(p_j) \subseteq p_j$, for any $j = 1, \ldots, r$, by a result of Seidenberg [Se]. We may also invoke once more the fact that $(f_1, \ldots, f_n)$ is a regular sequence to see that f_n represents a nonzero element $\bar{f}_n^{(j)}$ of $\mathbb{F}[x]/p_j$, $j = 1, \ldots, r$ (see, e.g., [Ma, pp. 50, 54, 95]). Thus, D will induce a derivation $D_j \in \mathrm{Der}^*(\mathbb{F}[x]/p_j)$, vanishing on $\bar{f}_n^{(j)}$ by our last hypothesis, $Df_n \in J$, $j = 1, \ldots, r$. It is therefore enough to prove that $D_j = 0$ for any j to conclude that $D \equiv 0$ modulo $\bigcap_{j=1}^r p_j = \sqrt{J}$, as asserted.

To this end we may thus suppose that Z is irreducible (and drop the index j from notations). Pick a nonzero point $z = (z_1, \ldots, z_n) \in Z$ and remark that the map $\tau: \mathbb{F} \to Z$ sending $t \in \mathbb{F}$ to $(t^{w_1}z_1, \ldots, t^{w_n}z_n) \in Z$ induces a graded algebra isomorphism $\tau^*: \mathbb{F}[x]/p \xrightarrow{\sim} \mathbb{F}[t^{w_i}/i \in S] \subseteq \mathbb{F}[t]$, $(|t| = 1)$ (by irreducibility and a straightforward dimension argument), where $S = \{i/z_i \neq 0\}$. In order to finish our proof we are thus led to prove that if $D \in \mathrm{Der}^k(\mathbb{F}[t^{w_i}/i \in S])(k \in \mathbb{Z})$ vanishes on $t^{\sum_{i \in S} \alpha_i w_i}$ (where $\alpha_i \in \mathbb{N}$, any $i \in S$, and $\sum_{i \in S} \alpha_i > 0$) then necessarily $D = 0$.

To see this property of homogeneous derivations on monomial curves (compare to [K]) one may argue as follows. Let $y_i = t^{w_i}$ $(i \in S)$ and notice that D is uniquely determined by the values taken on the algebra generators, $Dy_i = c_i t^{w_i + k}$, where $c_i \in \mathbb{F}$, for $i \in S$. Pick positive integers d and a_i, $i \in S$, such that $a_i w_i = d$, for any i. Consequently $y_i^{a_i} = y_j^{a_j}$ for any $i, j \in S$, whence $a_i c_i = a_j c_j$, by taking D-derivations; therefore $c_i = c w_i$, $i \in S$, for some $c \in \mathbb{F}$.

Finally $0 = D(\prod_{i \in S} y_i^{\alpha_i}) = c(\sum_{i \in S} \alpha_i w_i) t^{k + \sum_{i \in S} \alpha_i w_i}$, which gives $c = 0$ and consequently $D = 0$, as needed.

2. EXAMPLES

Our examples start with the following simple construction, taken from Hamm's 1969 thesis, see [Lo, Example 1, p. 8]. Given the weights $\{w_i\}_{1 \le i \le n}$, pick positive integers $a_i\,(1 \le i \le n)$ and d such that $w_i a_i = d$, for any $i = 1, \ldots, n$. A continuous parameter is provided by a matrix of $\mathbb{F}$ coefficients $\gamma = (\gamma_{ij})_{i=1,\ldots,n-1,\, j=1,\ldots,n}$, giving rise to the sequence of weighted homogeneous degree d polynomials

$$f_i = \sum_{j=1}^{n} \gamma_{ij} x_j^{a_j}, \qquad \text{for } i = 1, \ldots, n-1. \tag{1}$$

The novelty is to also consider a second continuous parameter which will be an arbitrary polynomial of degree $kd\,(k \ge 1)$, $f_n \in \mathbb{F}[x_1, \ldots, x_n]$. Assume moreover that $rk(\gamma) = n - 1$. We may pick $\gamma_{nj} \in \mathbb{F}$ $(1 \le j \le n)$ such that the matrix $(\gamma_{ij})_{1 \le i, j \le n}$ is nonsingular and consider $f_n = (\sum_{j=1}^{n} \gamma_{nj} x_j^{a_j})^k$. Note that plainly $(f_1, \ldots, f_n)$ is a regular sequence, since the zero set of $(f_1, \ldots, f_n)$ reduces to the origin. These remarks may be used to get simultaneously that under this assumption $(f_1, \ldots, f_{n-1})$ will be a regular sequence and that for such a fixed choice of γ

$$\mathscr{U} = \left\{ f_n \in \mathbb{F}[x_1, \ldots, x_n]^{kd} \mid (f_1, \ldots, f_{n-1}, f_n) \text{ is regular} \right\} \tag{2}$$

defines a Zariski open nonvoid subset of the second space of parameters. Indeed $(f_1, \ldots, f_n)$ is regular if and only if f_n does not vanish identically on any irreducible component Z_j of the zero set of $(f_1, \ldots, f_{n-1})$, as we have seen in the previous proof. By homogeneity this in turn is equivalent to the Zariski open condition "$f_n(z_j) \neq 0$, any j," where $z_j \in Z_j$ is a choice of a nonzero point.

Set then $J = (f_1, \ldots, f_{n-1})$, $B = \mathbb{F}[x]/J$, and $A = \mathbb{F}[x]/_{J+(f_n)}$, assuming from now on that the rank of γ is maximal. To get our reduced examples, we may start in fact with a quite general (weighted) homogeneous regular sequence $(f_1, \ldots, f_{n-1})$ in $\mathbb{F}[x]$ and perform the change of variable $x_i = z_i^{w_i}$, $1 \le i \le n$ (here $|z_1| = \cdots = |z_n| = 1$) to get a homogeneous regular sequence $(\bar{f}_1, \ldots, \bar{f}_{n-1})$ in $\mathbb{F}[z]$ (where $\bar{f}_i = f_i(z)$). One may use for example [Ma, Sections 4, 5, 13, and 16], the main point being that the regularity property is preserved by extension of ideals in the case of a faithfully flat extension of polynomial rings and in our case it is immediate to see that $\mathbb{F}[z]$ is in fact a free $\mathbb{F}[x]$-module. Setting $\bar{J} = (\bar{f}_1, \ldots, \bar{f}_{n-1})$ and $\bar{B} = \mathbb{F}[z]/\bar{J}$, it will thus suffice to see that $\bar{B}$ is reduced. Indeed $\bar{B}$ is reduced if and only if $\bar{J} = \bigcap \overline{p_i}$, with $\overline{p_i}$ are prime ideals in $\mathbb{F}[z]$, see [ZS, p. 209]. This implies that $\bar{J} \cap \mathbb{F}[x] = \bigcap (\overline{p_i} \cap \mathbb{F}[x])$, where the ideals $p_i = \overline{p_i} \cap \mathbb{F}[x]$ are again prime ideals in $\mathbb{F}[x]$. On the other hand $J = \bar{J} \cap \mathbb{F}[x]$ (by faithful flatness again [Ma, pp. 27, 28]). By [ZS, p. 209], again we may infer that B is reduced. Now in the homogeneous case one may use the geometric generalized Bézout test below (2.1 (i)), which is certainly well known. The reason for which we have also included a proofsketch was the lack of an adequate reference.

2.1. LEMMA. *Let $(f_1, \ldots, f_{n-1})$ be an arbitrary weighted homogeneous regular sequence. With the above notation we then have*:

(i) $\bar{B}$ *is reduced if and only if the number of solutions in projective $(n-1)$-space (counted without multiplicities!) of the equations $\bar{f}_1(z) = \cdots = \bar{f}_{n-1}(z) = 0$ equals* $\prod_{i=1}^{n-1} \deg(\bar{f}_i)$.

(ii) *In the particular case given by the formula* (1) (*where rank* $(\gamma) = n-1$), $\bar{B}$ *is reduced if and only if any nonzero solution $v = (v_1, \ldots, v_n)$ of the linear system*

$$\sum_{j=1}^{n} \gamma_{ij} v_j = 0, \qquad 1 \leq i \leq n-1, \tag{3}$$

has $v_i \neq 0$ for any $i = 1, \ldots, n$.

Proof. For (i) one may start by following the lines of the proof of the generalized Bézout theorem in $\mathbb{P}^{n-1}$ as in [Ha, I7.7 and I7.8], i.e., to compute the Hilbert polynomial of $\mathbb{F}[z]/\bar{J}$ in two ways. Using [Ha, I7.4 and I7.5], and keeping in mind that we are dealing with a complete intersection (in particular all the associated primes of $\bar{J}$ are minimal and one-dimensional, see [Ma, pp. 108–111]), one arrives at the formula

$$\prod_{i=1}^{n-1} \deg\left(\bar{f}_i\right) = \sum_{j=1}^{m} \operatorname{length}_{\mathbb{F}[z]_{p_j}}\left(\mathbb{F}[z]/\bar{J}\right)_{p_j}, \tag{4}$$

where $\bar{J} = \bigcap_{j=1}^{m} q_j$ is the primary decomposition and $\operatorname{Ass}(\bar{J}) = \{p_1, \ldots, p_m\}$. Consequently the number of distinct solutions in $\mathbb{P}^{n-1}$ of the system $\bar{f}_1(z) = \cdots = \bar{f}_{n-1}(z) = 0$ equals m. Next one knows [Ha, I7.4] that for all the multiplicities $\mu_j = \operatorname{length}_{\mathbb{F}[z]_{p_j}}(\mathbb{F}[z]/\bar{J})_{p_j} \geq 1$. Hence the first claim of the lemma will follow at once from

$$\mu_j = 1 \Leftrightarrow q_j = p_j, \qquad \text{for any } j, \tag{5}$$

given (4) and the already mentioned characterization of radical ideals [ZS, p. 209].

The assertion (5) is in turn a consequence of standard localization arguments. Firstly the exact sequence $0 \to \mathbb{F}[z]/\bar{J} \to \mathbb{F}[z]/q_j \oplus \mathbb{F}[z]/\bigcap_{k \neq j} q_k$, localized at p_j, provides an isomorphism $(\mathbb{F}[z]/\bar{J})_{p_j} =$

$(\mathbb{F}[z]/q_j)_{p_j}$, due to the fact that $\bigcap_{k \neq j} q_k \not\subseteq p_j$ (see [ZS, p. 210]). Hence $\mu_j = 1 \Leftrightarrow (p_j/q_j)_{p_j} = 0$ (since plainly $\text{length}_{\mathbb{F}[z]_p}(\mathbb{F}[z]/p)_p = 1$ for any prime p), see, e.g., [Ha, p. 51]. Finally $(p_j/q_j)_{p_j} = 0 \Leftrightarrow q_j = p_j$ (by using the definitions and the characterization of the zero divisors of $\mathbb{F}[z]/q_j$, see [Ma, pp. 50, 53]).

Part (ii) is a direct application of the preceding one. It will suffice to verify the assertion for a fixed nonzero solution v of the maximal rank linear system (3). Put then $m = \#\{i \mid v_i \neq 0\}$ and write $v_i = u_i^d$ for some $u_i \in \mathbb{F}$, for any i. It is then straightforward to see that the nonzero solutions z of the system $\bar{f}_1(z) = \cdots = \bar{f}_{n-1}(z) = 0$ are of the form $z_i = 0$ if $v_i = 0$ and $z_i = u_i\theta_i$ with $\theta_i^d = 1$ if $v_i \neq 0$ (up to a nonzero factor); therefore there are precisely d^{m-1} solutions in $\mathbb{P}^{n-1}$ and $\overline{B}$ is reduced if and only if $m = n$, as asserted.

As a simple application we may offer the following class of (not necessarily reduced) examples: let A be given by $A = \mathbb{F}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$, where $f_1, \ldots, f_{n-1}$ are constructed as in (1) out of an arbitrary matrix γ of maximal rank and $f_n \in \mathbb{F}[x]^{kd}$ is such that $(f_1, \ldots, f_{n-1}, f_n)$ is a regular sequence.

2.2. PROPOSITION. *For any A as above* $\mathrm{Der}^{<0}(A) = 0$

Proof. We may first put the ideal $J = (f_1, \ldots, f_{n-1})$ in normal form, as follows. Denote by v a nonzero solution of the linear system (3) and set $\mathrm{supp}(v) = \{i \mid v_i \neq 0\}$. If γ' is another similar matrix of rank $n-1$, consider the associated ideal J' and a nonzero solution v', as above. Note that $(\mathbb{F}^*)^n$ acts on $\mathbb{F}[x_1, \ldots, x_n]$ by graded algebra automorphisms, the action of $g = (\lambda_1, \ldots, \lambda_n)$ on the free algebra generators being given by $g(x_i) = \lambda_i x_i, 1 \leq i \leq n$. It is then immediate to see that $g \in (\mathbb{F}^*)^n$ is such that $g(J) = J'$ if and only if there exists $\mu \in \mathbb{F}^*$ with the property

$$\lambda_i^{a_i} v_i' = \mu v_i, \qquad \text{for any } i. \tag{6}$$

In particular the existence of such an element g is equivalent to $\mathrm{supp}(v) = \mathrm{supp}(v')$. Therefore setting $m = \#\,\mathrm{supp}(v)$, if $m = n$ then $\mathbb{F}[x]/J$ is reduced by Lemma 2.1 (ii) and the discussion preceding it; hence Corollary (i) is available and we are done. If not we may invoke the above criterion to see that up to isomorphism we may suppose that our given algebra is of the form

$$A = \mathbb{F}[x_1, \ldots, x_n]/\left(x_1^{a_1} - x_2^{a_2}, \ldots, x_{m-1}^{a_{m-1}} - x_m^{a_m}, x_{m+1}^{a_{m+1}}, \ldots, x_n^{a_n}, f_n\right).$$

In particular it is fibered in the sense of [M, pp. 154, 158]. The algebraic base is the graded algebra $B = \bigotimes_{j=m+1}^{n} \mathbb{F}[x_j]/(x_j^{a_j})$, which is trivially fibered, see [M, p. 155], as a tensor product of zero-dimensional weighted complete intersections on one generator. Hence all the factors are without nontrivial strictly negative degree derivations, see the Introduction, and thus $\mathrm{Der}^{<0}(B) = 0$ by the main result of [M], see Theorem 1, p. 154 therein. The algebraic fiber is a weighted zero-dimensional complete intersection of the form

$$F^* = \mathbb{F}[x_1, \ldots, x_m] / \left(x_1^{a_1} - x_2^{a_2}, \ldots, x_{m-1}^{a_{m-1}} - x_m^{a_m}, \bar{f}_m \right)$$

(where $\bar{f}_m$ is the class of f_n modulo $(x_{m+1}, \ldots, x_n)$), see [M, p. 158]. Our Lemma (ii) and Corollary (i) apply to give $\mathrm{Der}^{<0}(F) = 0$, whence $\mathrm{Der}^{<0}(A) = 0$ by the main result of [M].

2.3. *Remark.* Many-parameter families of isomorphism types of algebras A with $\mathbb{F}[x]/J$ reduced may be explicitly produced along the above lines, for any $n > 2$, as follows. Pick up weights

$$1 \leq w_1 < w_2 < \cdots < w_n < 2w_1 \tag{7}$$

(for example $w_i = n + i, 1 \leq i \leq n$) and even exponents a_i such that $w_i a_i = d, 1 \leq i \leq n$. Fix a maximal rank coefficient matrix γ as in Lemma 2.1 (ii) to get $\mathbb{F}[x]/J$ reduced. Then we claim that for any $k > 1$ the isomorphism types of the graded algebras $\{A(f_n) = \mathbb{F}[x]/J + (f_n) \mid f_n \in \mathscr{U}\}$ (see (2) for the definition of $\mathscr{U}$) are in bijection with the quotient of a nonvoid Zariski open subset in projective space $U \subseteq \mathbb{P}(V)$ by a finite group G action, where $\dim U \geq (n-2)(n+1)/2$. Here $V = (\mathbb{F}[x]/J)^{kd}, U$ is the image of $\mathscr{U}$ by the canonical projection $\mathbb{F}[x]^{kd} \xrightarrow{\pi} V, G = \prod_{i=1}^n G_i \subseteq (\mathbb{F}^*)^n$, with $G_i = \{\text{the } a_i\text{-roots of } 1\}$, $1 \leq i \leq n$. The action of $g \in G$ on U is induced by the $(\mathbb{F}^*)^n$-action on the graded algebra $\mathbb{F}[x]$ described in the previous proof (use $g(f_i) = f_i$, any $i < n$). To see that $A(f_n) \sim A(f'_n)(f_n, f'_n \in \mathscr{U})$ if and only if there is $g \in G$ such that $g(\overline{\pi f_n}) = \overline{\pi f_n{}'}$ (where the bar denotes the class in $\mathbb{P}(V)$ of a nonzero vector of V), remark first that under the assumptions (7) the elements $h \in (\mathbb{F}^*)^n$ exhaust the graded algebra automorphisms of $\mathbb{F}[x]$. Next, $k > 1$ forces $h(J) = J$ assuming that h induces an isomorphism $A(f_n) \xrightarrow{\sim} A(f'_n)$. This in turn implies $h = tg$ with $g \in G$ and t a grading automorphism as in Corollary (ii) (use (6)). Therefore $h(J + (f_n)) = J + (f'_n)$ if and only if $g(\overline{\pi f_n}) = \overline{\pi f'_n}$ as asserted, since $g(f_i) = f_i$ for any $i < n$, as noted before. Finally the assertion on $\dim U$ follows from

$$\dim(\mathbb{F}[x]/J)^{kd} \geq \frac{n(n-1)}{2}. \tag{8}$$

To get the estimation (8) note that $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \le \alpha_i < a_i, 1 \le i \le n\}$ is a $\mathbb{F}[y]$-basis of $\mathbb{F}[x]$ (where $y_i = x_i^{a_i}, 1 \le i \le n$, in particular $|y_i| = d$, for any i), and hence the classes mod J of the above monomials will also provide a $\mathbb{F}[y]/J'$-basis of $\mathbb{F}[x]/J$. Here J equals the extension (see (1)) of the ideal $J' \subseteq \mathbb{F}[y_1, \dots, y_n]$ generated by $n-1$ independent linear relations, $f_i' = \sum_{j=1}^n \gamma_{ij} y_j, i < n$, whence $\mathbb{F}[y]/J' \sim \mathbb{F}[y_0]$, the free graded algebra on one degree d generator y_0. One may then consider the following $n(n-1)/2$ linearly independent elements of $(\mathbb{F}[x]/J)^{kd}$

$$\left\{x_i^{a_i/2}\, x_j^{a_j/2}\, y_0^{k-1} \mid 1 \le i < j \le n\right\}$$

to get the desired inequality.

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